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ELASTIC BEAMS OF GREATEST LATERAL EXTENT

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Abstract—Numerical solutions are presented for the deflections of slender, uniform elastic beams, self-loaded and built into a supporting wall. The beams are alike in every property but length. There exists a critical length which maximizes the lateral reach; longer beams droop so much under their own weight that their tips actually move closer to the supporting wall. The critical length depends only on the angle at which the beam leaves the supporting wall.

1. INTRODUCTION

SUPPOSE that a uniform cantilever beam is loaded under its own weight. The deflected shape of such an elastic beam has been considered by Holden [1] and Wang [2], each of whom employed numerical methods to calculate the locus of the elastic line for a range of values of the uniformly distributed load. An important question which has apparently not yet received attention concerns the properties of beams whose tips extend the greatest possible lateral distance away from the cantilever support. It will be shown that the thickness of such uniform beams, when made of the same material, must increase as the length raised to the $\frac{3}{2}$ power, a fact which has important consequences in determining the way the proportions of both plants and animals change with size.

2. THE UNIFORMLY LOADED CANTILEVER

In Fig. 1, a beam whose flexural rigidity is EI is loaded by a weight w per unit length. Length along the neutral axis is s and the tangent to the axis makes an angle θ with the horizontal. The beam makes an angle θ_o with the support at its built-in end. For such slender beams,

$$\frac{d^2\theta}{ds^2} = -\frac{w}{EI}(l-s)\cos\theta \tag{1}$$

with $\theta = \theta_o$ at $s = 0$, $d\theta/ds = 0$ at $s = l$.

Introducing a dimensionless arc length and load parameter as respectively

$$\eta = \frac{s}{l} \tag{2}$$

and

$$k = \frac{wl^3}{EI} \tag{3}$$

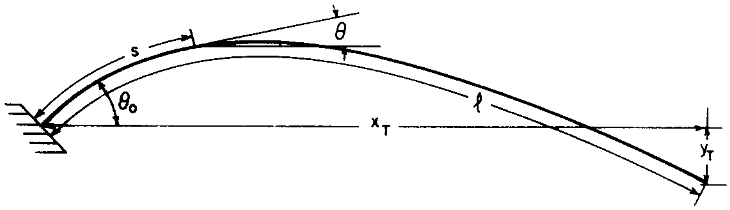

FIG. 1. Length along the neutral axis is s. The tangent to the neutral axis makes an angle θ with the horizontal. Flexural rigidity is EI and weight per length is w.

the equation becomes

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}\eta^2} = -k(1-\eta)\cos\theta \tag{4}$$

with $\theta = \theta_o$ at $\eta = 0$, $\mathrm{d}\theta/\mathrm{d}\eta = 0$ at $\eta = 1$.

Assuming some initial shape $\theta(\eta)$, (in this case the third-order polynomial given by linear theory), equation (4) may be solved numerically by calculating values for the right-hand side, integrating twice, and thereby obtaining a new $\theta(\eta)$ which may be iterated until the desired accuracy is achieved. The x and y coordinates of the neutral axis are then

$$x(\eta) = l\int_o^\eta \cos\theta\,\mathrm{d}\eta \tag{5}$$

and

$$y(\eta) = l\int_o^\eta \sin\theta\,\mathrm{d}\eta. \tag{6}$$

In the present work, Simpson's rule was employed for the integrations and the accuracy of approximation to the solution of (4) was $1{\cdot}0\times10^{-5}$.

3. LOCUS OF THE TIP

We are interested in the successive shapes of a long, thin cantilever beam as w and EI (hence diameter or thickness) is kept constant but the length l is increased.

Dimensionless coordinates for the neutral axis will be defined as

$$X = x\bigg/\left(\frac{EI}{w}\right)^{\frac{1}{3}} = k^{\frac{1}{3}}\int_o^\eta \cos\theta\,\mathrm{d}\eta \tag{7}$$

and

$$Y = y\bigg/\left(\frac{EI}{w}\right)^{\frac{1}{3}} = k^{\frac{1}{3}}\int_o^\eta \sin\theta\,\mathrm{d}\eta. \tag{8}$$

Figure 2(a) shows the elastic line of such a beam for two different lengths when the launch angle θ_o is 0. The locus of the tip of the beam ($\eta = 1$) as l changes is shown as a solid line. It is apparent that the tip of the beam reaches a maximum lateral extent X_{max} for a specific

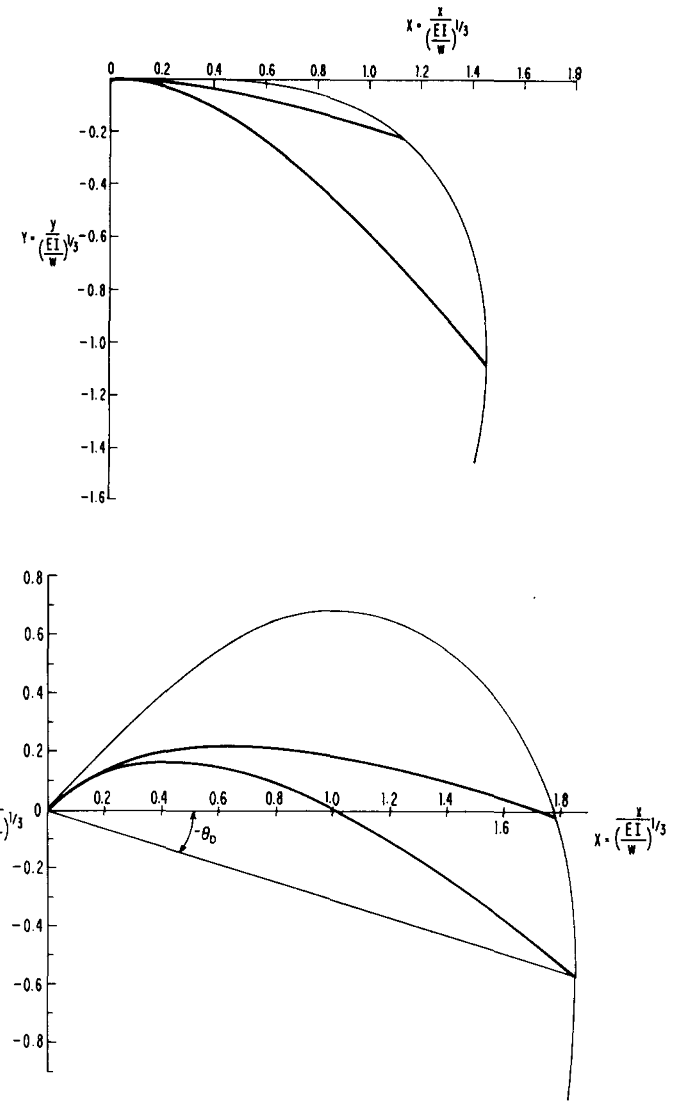

FIG. 2(a). Locus of the tip when $\theta_o = 0$. The shapes of the neutral axes for beams of two different lengths but otherwise the same properties are also shown. (b). Locus of the tip when $\theta_o = \pi/4$. The beam of greatest lateral extent has a chord which makes an angle $-\theta_D$ with the horizontal.

value of l; longer beams droop so much that their tips actually move closer to the supporting wall. A beam of this critical length has a chord which makes an angle θ_D with the horizontal, where θ_D will be termed the "droop angle". This θ_D depends only on the launch angle θ_o. Figure 2(b) illustrates how the locus of the tip and hence θ_D is changed when θ_o is $\pi/4$.

In Fig. 3, both X_{max} and the droop angle θ_D are plotted vs the launch angle θ_o for $-\pi/2 \leq \theta_o \leq \pi/2$. It is evident that X_{max} reaches its greatest value when θ_o is in the vicinity of $\pi/2$, so that a fisherman paying out a long, limber pole should ultimately hold his end

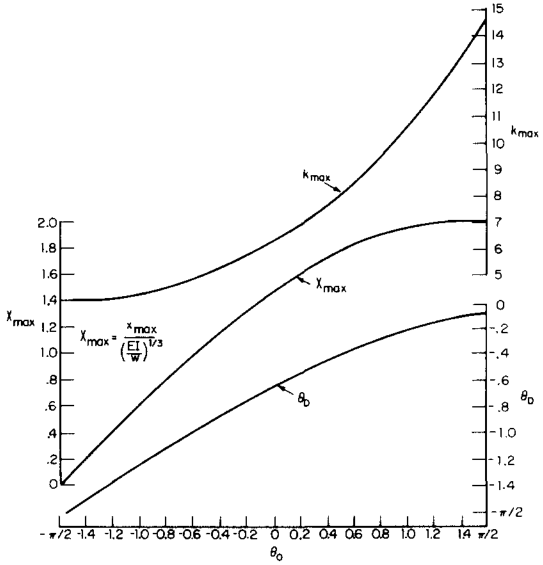

FIG. 3. Droop angle $-\theta_D$, maximum lateral extent X_{max}, and the corresponding load factor k_{max} as a function of launch angle θ_o. For any θ_o, k_{max} determines the length of the beam of greatest lateral extent.

of the pole straight up to assure that the tip reaches out as far as possible. The droop angle θ_D is equal to the launch angle θ_o when $\theta_o = -\pi/2$, but slowly approaches 0 as θ_o is increased. Notice that θ_D is still slightly negative as θ_o goes through $\pi/2$. A curve showing how k_{max}, the load parameter for the beam of greatest lateral extent, depends on θ_o is also shown. As expected, k_{max} increases with increasing θ_o.

4. CONCLUSIONS

A cantilever beam of the greatest lateral extent is one whose load parameter $k_{max}(\theta_o)$ maximizes the dimensionless displacement of the tip, $X(1)$. If such a beam has a rectangular cross-section, the critical length becomes

$$l_{cr} = \left[\frac{k_{max}}{12}\right]^{\frac{1}{3}}\left[\frac{E}{\rho g}\right]^{\frac{1}{3}} \mathrm{d}^{\frac{2}{3}} \tag{9}$$

where ρg is the weight/unit volume of the material, and d is the depth of the cross-section. For beams of circular or elliptical cross-section,

$$l_{cr} = \left[\frac{k_{max}}{16}\right]^{\frac{1}{3}}\left[\frac{E}{\rho g}\right]^{\frac{1}{3}} \mathrm{d}^{\frac{2}{3}} \tag{10}$$

where d is the diameter of the circular cross-section or the diameter of the ellipse in the plane of bending.

An important conclusion may be inferred from equations (9) and (10), which differ only by a numerical constant. In comparing elastic beams of the greatest lateral extent made of the same material with the same cross-sectional shape, one should find that length increases as thickness or diameter raised to the $\frac{2}{3}$ power. McMahon [3] has examined the consequences of this prediction, when combined with an identical prediction based on the elastic stability of slender columns under their own weight, in determining limits for the proportions of trees and animals. A model requiring the lengths of living organisms to increase as the diameter to the $\frac{2}{3}$ power is found to be in agreement with published data, arguing that elastic criteria impose limits on biological proportions, and consequently, as derived in [3], on metabolic rates.

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Абстракт—Даются численные решения для прогибов гибких, однородно упругих балок, самонагруженных и заделанных в поддерживающей стенке. Балки в каждом свойстве одинаковы, за исключением длины. Находится некоторая критическая длина, которая увеличивает до крайности поперечный вынос стрелы прогиба. Более длниные балки склоняются тем значительно под влиянием собственного веса, что их концы фактически перемещаются ближе к поддерживающей стенке. Критическая длина зависит только от угла, при котором балка оставляет поддерживающую стенку.